ASYMPTOTIC COVERAGE DISTRIBUTIONS ON THE CIRCLE

BY

ANDREW F. SIEGEL

TECHNICAL REPORT NO. 17
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Andrew F. Siegel

1. Introduction and Summary.

Coverage problems arise in a wide variety of applications. The particular problem of the coverage of a circle by n random equal arcs, each of length a_n, has been associated with research in immunology (Moran and Fazekas de St. Groth, 1962), genetics (Stevens, 1939), and time series analysis (Fisher, 1940). In a previous paper (Siegel, 1978a), exact formulae for the moments and distribution of coverage were obtained for this problem. The purpose of the present paper is to explore the asymptotic behavior of the coverage distribution for large n, because the exact distributions become difficult to evaluate numerically in this case. It is hoped that these limiting distributions will prove to be useful approximations in more general coverage problems.

Definitions, notation, and a distributional representation of the coverage are given in section 2. The asymptotic distribution of the vacancy is found in sections 3 and 4, under different behavior of the arc length sequence. In section 3, the arc length is chosen so that the probability of complete coverage of the circle remains fixed as n grows. The limiting distribution is found to be a mixture of a discrete mass point at zero with a positive continuous random variable, and may be interpreted in a natural way as a noncentral chi-square distribution with zero degrees of freedom. Section 4 treats proportionately smaller arc lengths, and a limiting normal distribution is obtained.

2. Definitions, Notation, and a Distributional Representation of the Coverage.

Let n arcs, each of length a, be placed independently with centers uniformly distributed on the edge of a circle of circumference one. Denote these arcs by $A_1, \ldots, A_n$. The coverage is defined as

$$C(n,a) = \mu\left(\bigcup_{i=1}^{n} A_i\right) \tag{2.1}$$

where μ denotes Lebesgue measure on the circle. $C(n,a)$ is the (random) proportion of the circle that is contained in at least one arc. The vacancy is defined as

$$V(n,a) = 1-C(n,a) \tag{2.2}$$

and is the random proportion of the circle that is contained in no arc. It is introduced because it is generally easier to work with mathematically than is the coverage. For a thorough treatment of the foundations of coverage problems, the reader is referred to Ailam (1966).

The event $V(n,a) = 0$ represents complete coverage of the circle by these arcs. Its probability will be denoted by $P(n,a)$, and was found by Stevens (1939) to be

$$P(n,a) = P(V(n,a)=0) = \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n}{\ell} (1-\ell a)_{+}^{n-1} \tag{2.3}$$

where $(t)_+ = \max(t,0)$.

R. A. Fisher (1940) discovered a link between this coverage problem and the analysis of time series, from which we obtain a useful representation of the vacancy. Let $X_1, \dots, X_n$ be independent and identically distributed χ^2_2 (or alternatively, any exponential distribution). Normalize them so that they sum to one by defining

$$Y_i = X_i / \sum_{j=1}^{n} X_j \ . \tag{2.4}$$

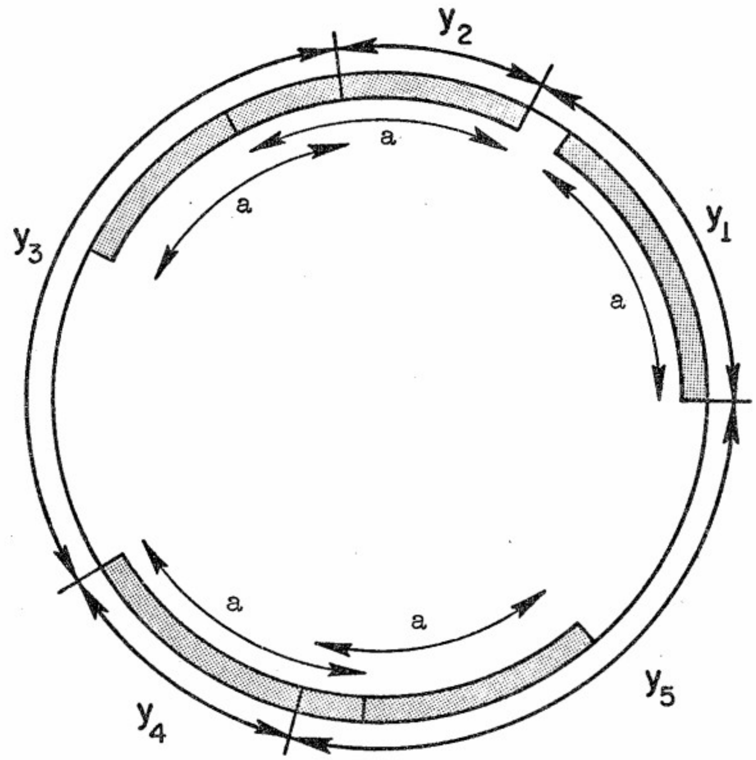
Then a distributional representation of the vacancy is given by

$$V(n,a) = \sum_{i=1}^{n} (Y_i - a)_+ \ . \tag{2.5}$$

This follows because the Y_i may be interpreted as the spacings between adjacent counter-clockwise endpoints of arcs, as is illustrated in figure 2.1.

Figure 2.1

$Y_1,\ldots,Y_n$ generate n random arcs of size a on the circle, in the case $n=5$.



3. Limiting Coverage Distribution: Constant Coverage Probability.

In this section we find the limiting distribution of the vacancy in the case in which the coverage probability stays constant. Thus for an experiment in which n random arcs are placed, the length of each arc will be a_n and is chosen so that $P(n,a_n) = \gamma$, where γ is the fixed coverage probability and lies strictly between zero and one. It is important to note that in a given experiment, all arcs placed are of the same length. The treatment of random arcs of different sizes is considerably more complicated (Siegel, 1978b).

In theorem 3.1, the behavior of the sequence a_n is characterized, and in theorem 3.2, the asymptotic distribution of the vacancy is found. Following this is a discussion of the interpretation of this distribution as a noncentral chi-square distribution with zero degrees of freedom.

Theorem 3.1. Let $\beta = \log(1/\gamma)$. Then

$$\lim_{n \to \infty} P(n, \frac{1}{n}\log(\frac{n}{\beta})) = \gamma \tag{3.1}$$

and

$$\lim_{n \to \infty} n[a_n - \frac{1}{n}\log(\frac{n}{\beta})] = 0 \tag{3.2}$$

where, we recall, a_n satisfies $P(n,a_n) = \gamma$.

Note that (3.2) is a much stronger statement than $a_n \sim \frac{1}{n}\log(\frac{n}{\beta})$, meaning their ratio tends to one in the limit. A stronger result is clearly needed because $\frac{1}{n}\log(\frac{n}{\beta_1}) \sim \frac{1}{n}\log(\frac{n}{\beta_2})$ for any positive β_1 and β_2.

Proof. Let $G_n = \max_{1 \le i \le n} Y_i$, using the representation in section 2. The circle is covered by n arcs of length $\frac{1}{n}\log(\frac{n}{\beta})$ if and only if $G_n \le \frac{1}{n}\log(\frac{n}{\beta})$. Barton and David (1956) showed that

$$(3.3) \qquad 2n \exp(-nG_n) \xrightarrow{\mathcal{D}} \chi_2^2 .$$

Hence

$$(3.4) \qquad P(n, \tfrac{1}{n}\log(\tfrac{n}{\beta})) = P(2n \exp(-nG_n) \ge 2\beta) \to e^{-\beta} = \gamma$$

and (3.1) is established. For (3.2), we choose β_1 and β_2 that satisfy $0 < \beta_1 < \beta < \beta_2$. Then $\gamma_2 = \exp(-\beta_2) < \gamma < \exp(-\beta_1) = \gamma_1$ and therefore, for sufficiently large n we have

$$(3.5) \qquad P(n, \tfrac{1}{n}\log(\tfrac{n}{\beta_2})) < P(n,a_n) < P(n, \tfrac{1}{n}\log(\tfrac{n}{\beta_1}))$$

using the convergence result (3.1) just established. But $P(n,t)$ is an increasing function of t, so

$$(3.6) \qquad \tfrac{1}{n}\log(\tfrac{n}{\beta_2}) < a_n < \tfrac{1}{n}\log(\tfrac{n}{\beta_1})$$

and therefore (for sufficiently large n)

$$(3.7) \qquad \log(\tfrac{\beta}{\beta_2}) < n[a_n - \tfrac{1}{n}\log(\tfrac{n}{\beta})] < \log(\tfrac{\beta}{\beta_1}) .$$

By choosing β_1 and β_2 close to β, we can make the outer terms in (3.7) as near to zero as we wish, completing the proof. ||

Theorem 3.2. The limiting distribution of the vacancy $V(n,a_n)$, where the arc lengths a_n are chosen so that the coverage probability remains fixed at γ, is given by

$$(3.8) \qquad nV(n,a_n) \xrightarrow{\mathcal{D}} Y$$

where Y is the mixture

$$(3.9) \qquad Y = \begin{cases} 0 & \text{probability} \quad \gamma \\ Z & \text{probability} \quad 1-\gamma \end{cases}$$

and Z is continuous with density

$$(3.10) \qquad f_\beta(t) = \frac{1}{e^\beta - 1} \sum_{\ell=1}^{\infty} \frac{\beta^\ell}{\ell!} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t}$$

$$= \frac{\sqrt{\beta}}{e^\beta - 1} \cdot \frac{e^{-t}}{\sqrt{t}} I_1(2\sqrt{\beta t})$$

where we recall $\beta = \log(1/\gamma)$, and I_1 denotes a modified Bessel function (see, for example, chapter 9 of Oliver, 1964). The cumulative distribution function of Y is

$$(3.11) \qquad F_\beta(t) = P(Y \leq t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} [e^{-\beta} \sum_{\ell=0}^{k} \frac{\beta^\ell}{\ell!}]$$

where $0 \leq t < \infty$.

Proof. Set

$$(3.12) \qquad b_n = \frac{1}{n}\log\left(\frac{n}{\beta}\right) .$$

We will use $V(n,b_n)$ as an approximation to $V(n,a_n)$. Moments of vacancy were found by Siegel (1978a), and are

$$(3.13) \qquad E[nV(n,b_n)]^m = n^m\binom{m+n-1}{n}^{-1} \sum_{\ell=1}^{m} \binom{m}{\ell}\binom{n-1}{\ell-1}\left(1-\frac{\ell}{n}\log\left(\frac{n}{\beta}\right)\right)_+^{m+n-1} .$$

It may be verified that

$$(3.14) \qquad \lim_{n\to\infty} \left(1-\frac{\ell}{n}\log\frac{n}{\beta}\right)^{m+n-1} = \left(\frac{\beta}{n}\right)^{\ell}$$

by taking logs and doing a Taylor series expansion. Using this, the limit of (3.13) may be calculated, and we define

$$(3.15) \qquad \mu_m = \lim_{n\to\infty} E[nV(n,b_n)]^m = m! \sum_{\ell=1}^{m} \binom{m-1}{\ell-1}\frac{\beta^\ell}{\ell!} .$$

The moments of Y may also be calculated:

$$(3.16) \qquad EY^m = (1-\gamma)\int_0^\infty t^m f_\beta(t)dt = e^{-\beta}\sum_{\ell=1}^{\infty}\frac{\beta^\ell}{\ell!(\ell-1)!}\int_0^\infty t^{m+\ell-1}e^{-t}dt$$

where we use the monotone convergence theorem in order to exchange sum and integral. The integral is easily done, and we obtain

$$EY^m = e^{-\beta} \sum_{\ell=1}^{\infty} \frac{\beta^\ell}{\ell!(\ell-1)!} (m+\ell-1)! \; . \tag{3.17}$$

Now if we expand $e^{-\beta}$, multiply the two series, and gather powers of β, we get

$$EY^m = m! \sum_{\ell=1}^{\infty} \Big[\sum_{k=1}^{\ell} (-1)^{k+\ell} \binom{m+k-1}{k-1} \binom{\ell}{k} \Big] \frac{\beta^\ell}{\ell!} \; . \tag{3.18}$$

The term in brackets is $\binom{m-1}{\ell-1}$; to see this, simply expand the identity $(1+t)^\ell (1-t)^{-(m+1)} = (1-t)^{-(m-\ell+1)}$ and equate coefficients of $t^{\ell-1}$. Thus, comparing with (3.15) we have

$$EY^m = \mu_m = \lim_{n \to \infty} E[nV(n,b_n)]^m \; . \tag{3.19}$$

Convergence of moments implies convergence in distribution provided

$$\overline{\lim_{m \to \infty}} \; \frac{|\mu_m|^{1/m}}{m} < \infty \tag{3.20}$$

(see section 8.12 of Brieman, 1968). To establish (3.20), we use Stirling's formula:

$$(3.21) \qquad \mu_m = m! \sum_{\ell=1}^{m} \binom{m-1}{\ell-1} \frac{\beta^\ell}{\ell!} \leq m! \binom{m}{m/2} e^\beta \sim 2^{m+1} e^\beta m^m e^{-m}$$

from which (3.20) now follows easily. This proves that

$$(3.22) \qquad nV(n,b_n) \xrightarrow{\mathcal{L}} Y \text{ .}$$

To now prove (3.8), it will suffice to show that

$$(3.23) \qquad W_n = n[V(n,a_n)-V(n,b_n)] \xrightarrow{P} 0 \text{ .}$$

From theorem 3.1, $a_n = b_n + \frac{c_n}{n}$ where $c_n \to 0$. Using this and the representation of section 2, we have

$$(3.24) \qquad P(|W_n| > \varepsilon) = P(n|\sum_{i=1}^{n} \{(Y_i - b_n - \frac{c_n}{n})_+ - (Y_i - b_n)_+\}| > \varepsilon)$$

$$\leq P(|c_n| \sum_{i=1}^{n} I\{Y_i > b_n - \frac{|c_n|}{n}\} > \varepsilon)$$

where $I\{A\}$ is 1 if A holds and 0 otherwise. Applying the Markov inequality to (3.24), we have

$$(3.25) \qquad P(|W_n| > \varepsilon) \leq \frac{|c_n|}{\varepsilon} [nP(Y_1 > b_n - \frac{|c_n|}{n})] \text{ .}$$

It can be verified directly that if $0 \leq t \leq 1$, then

$$(3.26) \qquad P(Y_1 > t) = (1-t)^{n-1} \text{ .}$$

Using this, we now show that the bracketed term of (3.25) is bounded. This term is

$$(3.27) \qquad nP(Y_1 > b_n - \frac{|c_n|}{n}) = n(1 - b_n + \frac{|c_n|}{n})^{n-1} \quad .$$

Taking logs and expanding, we have

$$(3.28) \qquad \log(n) + (n-1)\log(1 - b_n + \frac{|c_n|}{n})$$

$$= \log(n) - (n-1)\{\frac{1}{n}\log(\frac{n}{\beta}) - \frac{|c_n|}{n} + O(\frac{\log n}{n})^2\}$$

$$= \log(\beta) + o(1) \; .$$

Using this in (3.25) and recalling that $c_n \to 0$, we have

$$(3.29) \qquad P(|W_n| > \varepsilon) \leq \frac{|c_n|}{\varepsilon} O(1) \to 0$$

completing the proof of (3.8). (3.11) follows from term-by-term integration of (3.10). ||

It is interesting to interpret the limiting distribution $2Y$ of $2n\,V(n,a_n)$ of theorem 3.2 as $\chi_0^2(\beta)$, a noncentral chi-square with zero (!) degrees of freedom and noncentrality parameter β. We see from (3.9) and (3.10) that $2Y$ is a Poisson mixture of the central chi-square deviates χ_{2K}^2 where $K \sim P_0(\beta)$, where we use the convention that χ_0^2 denotes a point mass at zero (this Dirac delta distribution is, in fact, the limit of χ_ν^2 as we let ν decrease to zero.) This representation easily handles the dual discrete-continuous nature of $2Y$.

Recall the representation of $\chi^2_\nu(\beta)$, ν a positive integer, as a Poisson mixture of the central chi-squares $\chi^2_{\nu+2K}$ where $K \sim P_0(\beta)$; see, for example, chapter 2.4 of Searle (1971). Thus we see by analogy that the limiting distribution of $2n\,V(n,a_n)$ naturally extends this definition to the case $\nu = 0$. Having zero degrees of freedom allows a mixture of a discrete mass point at zero (which corresponds to the event of complete coverage of the circle) with continuous variation (which corresponds to partial coverage of the circle.)

4. Proportionately Smaller Arcs.

The previous section treated the case of constant coverage probability, in which the arc size behaved like $\frac{1}{n}\log(\frac{n}{\beta})$. Now we consider proportionately smaller arcs, of length

$$d_n = \frac{\lambda}{n}\log(\frac{n}{\beta}) \tag{4.1}$$

where $0 < \lambda < 1$. Because in this case the coverage probability $P(n,d_n)$ tends to zero, as may be verified from (3.1), there are no mass points in the limiting distribution of the vacancy. The main result of this section is

Theorem 4.1. The vacancy $V(n,d_n)$ is asymptotically normal with mean $(\beta/n)^\lambda$ and variance $2\beta^\lambda n^{-(1+\lambda)}$. That is,

$$(4.2) \qquad \frac{V(n,d_n)-(\beta/n)^\lambda}{\sqrt{2\beta^\lambda\ n^{-(1+\lambda)}}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)\ .$$

Proof. We will use

$$(4.3) \qquad V_n^* = \sum_{i=1}^{n} \left(\frac{X_i}{2n} - d_n\right)_+$$

as an approximation to

$$(4.4) \qquad V(n,d_n) = \sum_{i=1}^{n} \left(X_i/\left(\sum_{j=1}^{n} X_j\right)-d_n\right)_+$$

where $X_1,\ldots,X_n$ are independent and identically distributed χ_2^2, from the representation of section 2. Since V_n^* is the sample mean of shifted and censored exponentials, it is easily seen that

$$(4.5) \qquad E(V_n^*) = (\beta/n)^\lambda$$

and

$$(4.6) \qquad \mathrm{Var}(V_n^*) = 2\beta^\lambda\ n^{-(1+\lambda)}[1 - \tfrac{1}{2}\ (\beta/n)^\lambda]\ .$$

Thus the mean and variance of V_n^* and $V(n,d_n)$ from (4.2) are asymptotically identical. The proof of asymptotic normality will follow immediately from the following two lemmas. The first will show that V_n^* is asymptotically normal with the right mean and variance, and the second

lemma will show that V_n^* is close enough to $V(n,d_n)$ to imply (4.2).

Lemma 4.1. For V_n^* defined in (4.3) and (4.1),

$$(4.7) \qquad \frac{V_n^* - (\beta/n)^\lambda}{\sqrt{2\beta^\lambda n^{-(1+\lambda)}}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) .$$

Proof. It will suffice to verify the Lindeberg condition for triangular arrays, which may be found in Loève (1960). This requires that

$$(4.8) \qquad g_n(\varepsilon) = nE\, Z_n^2\, I\{\,|Z_n| \geq \varepsilon\} \to 0$$

hold for each $\varepsilon > 0$, where

$$(4.9) \qquad Z_n = [\,(\frac{X_1}{2n} - d_n)_+ - \frac{1}{n}(\frac{\beta}{n})^\lambda] \cdot [\frac{2}{n}(\frac{\beta}{n})^\lambda (1 - \frac{1}{2}(\frac{\beta}{n})^\lambda)]^{-1/2}$$

is the first term, $(\frac{X_1}{2n} - d_n)_+$, in the sum for V_n^*, normalized to have mean zero and variance $1/n$. To establish (4.8), we bound $g_n(\varepsilon)$ using the fourth moment:

$$(4.10) \qquad g_n(\varepsilon) \leq \frac{n}{\varepsilon^2} E\, Z_n^4$$

$$= \frac{n^{3+2\lambda}}{4\varepsilon^2\beta^{2\lambda}[1 - \frac{1}{2n}(\beta/n)^\lambda]^2} E[\,(\frac{X_1}{2n} - d_n)_+ - \frac{1}{n}(\frac{\beta}{n})^\lambda\,]^4 .$$

We do a binomial expansion of the expectation term and calculate the moments from

(4.11) $$E\left(\frac{X_1}{2n} - d_n\right)_+^m = \frac{m!}{n^m}\,(\beta/n)^{\lambda} \quad , \quad m \geq 1 \,.$$

Thus (4.10) becomes

(4.12) $$g_n(\varepsilon) \leq O(1) \cdot \left\{\frac{24\beta^{\lambda}}{n^{4+\lambda}} - \frac{24\beta^{2\lambda}}{n^{4+2\lambda}} + \frac{12\beta^{3\lambda}}{n^{4+3\lambda}} - \frac{3\beta^{4\lambda}}{n^{4+4\lambda}}\right\}$$

$$= O\left(\frac{1}{n^{1-\lambda}}\right)$$

which completes the proof. ||

Lemma 4.2. Using the notation of this section,

(4.13) $$\frac{V(n,d_n) - V_n^*}{\sqrt{n^{-(1+\lambda)}}} \xrightarrow{P} 0 \,.$$

Proof. Using the representations (4.3) and (4.4), we must show that

(4.14) $$P\left(\left|\sum_{i=1}^{n}\left(\frac{X_i}{n\bar{X}_n} - d_n\right)_+ - \left(\frac{X_i}{2n} - d_n\right)_+\right| > \varepsilon n^{-(1+\lambda)/2}\right) \to 0$$

as $n \to \infty$, for each $\varepsilon > 0$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. For convenience, let B_n denote the event in (4.14). Because

(4.15) $$\sqrt{n}\left(\frac{2}{\bar{X}_n} - 1\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \,,$$

given $\delta > 0$ we can find K such that

(4.16) $$P\left(0 < \frac{2}{\bar{X}_n} - 1 < \frac{K}{\sqrt{n}}\right) > \frac{1}{2}(1-\delta)$$

and

$$(4.17) \qquad P\left(-\frac{K}{\sqrt{n}} < \frac{2}{\bar{X}_n} - 1 < 0\right) > \frac{1}{2}(1-\delta)$$

wherever n is sufficiently large. Then $P(B_n)$, from (4.14), satisfies

$$(4.18) \qquad P(B_n) \leq P\left(B_n, \left(1+\frac{K}{\sqrt{n}}\right)^{-1} < \frac{\bar{X}_n}{2} < 1\right) + P\left(B_n, 1 < \frac{\bar{X}_n}{2} < \left(1-\frac{K}{\sqrt{n}}\right)^{-1}\right) + \delta .$$

We consider the two main terms in the right hand side of (4.18). The first one is

$$(4.19) \qquad T_1 = P\left(\sum_{i=1}^{n} \left(\frac{X_i}{n\bar{X}_n} - d_n\right)_+ - \left(\frac{X_i}{2n} - d_n\right)_+ > \varepsilon n^{-(1+\lambda)/2}, \left(1+\frac{K}{\sqrt{n}}\right)^{-1} < \frac{\bar{X}_n}{2} < 1\right) .$$

The largest any single difference can be in the above sum is $KX_i/2n^{3/2}$, and this difference will be zero whenever $X_i \leq \dfrac{2nd_n}{1+K/\sqrt{n}}$. Thus

$$(4.20) \qquad T_1 \leq P\left(\sum_{i=1}^{n} X_i I\{X_i > \frac{2nd_n}{1+K/\sqrt{n}}\} > \frac{2\varepsilon}{K} n^{1-\lambda/2}\right) .$$

The Markov inequality yields

$$(4.21) \qquad T_1 \leq \frac{Kn^{\lambda/2}}{2\varepsilon} E\, X_1 I\{X_1 > \frac{2nd_n}{1+K/\sqrt{n}}\} .$$

This expectation is straightforward to calculate, and is

$$(4.22) \qquad 2\left\{1 + \frac{nd_n}{1+K/\sqrt{n}}\right\}\left(\frac{n}{\beta}\right)^{-\lambda/(1+K/\sqrt{n})} \leq 2\{1+nd_n\}\left(\frac{n}{\beta}\right)^{-(1/2+\zeta)\lambda}$$

for sufficiently large n, where ζ is any number strictly between 0 and 1/2. Thus the first term of (4.18) satisfies

$$(4.23) \qquad T_1 \le \frac{Kn^{\lambda/2}}{\varepsilon} \{1 + \lambda \log(\tfrac{n}{\beta})\} (\tfrac{n}{\beta})^{-(1/2+\zeta)\lambda} = O(n^{-\zeta\lambda} \log n) .$$

The second main term, T_2, of the right hand side of (4.18) may be treated similarly to obtain

$$(4.24) \qquad T_2 = O(n^{-\zeta\lambda} \log n)$$

as well. Thus

$$(4.25) \qquad P(B_n) = \delta + O(n^{-\zeta\lambda} \log n)$$

holds for all $\delta > 0$ and $\zeta \in (0,1/2)$, completing the proof. ||

Because convergence in distribution in theorem 4.1 does not, by itself, imply proper behavior of the moments of $V(n,d_n)$, this is treated in the following theorem.

Theorem 4.2. The asymptotic mean and variance of $V(n,d_n)$ are

$$(4.26) \qquad \mu_n = E\, V(n,d_n) \sim (\beta/n)^{\lambda}$$

and

$$(4.27) \qquad \sigma_n^2 = \mathrm{Var}(V(n,d_n)) \sim 2\beta^{\lambda}/n^{1+\lambda}.$$

Proof. Exact formulae for the moments are available in Siegel (1978a). The first moment is

$$(4.28) \qquad \mu_n = (1-d_n)^n = (1 - \tfrac{\lambda}{n} \log(\tfrac{n}{\beta}))^n .$$

Expanding $\log(\mu_n)$ in a Taylor series, we have

$$(4.29) \qquad \log(\mu_n) = \log(\beta/n)^{\lambda} + O(\tfrac{1}{n} \log(\tfrac{n}{\beta})^2)$$

which proves (4.26). The exact formula for the variance is

$$(4.30) \qquad \sigma_n^2 = \frac{2}{n+1} (1-d_n)^{n+1} + \frac{n-1}{n+1} (1-2d_n)^{n+1} - (1-d_n)^{2n} .$$

The first of the three terms on the right side of (4.30) is $\sim 2\beta^{\lambda}/n^{1+\lambda}$, using the same expansion technique we just used for μ_n. To complete the proof, we will show that the remaining two terms combined are $o(1/n^{1+\lambda})$. Write the second term from (4.30) as

$$(4.31) \qquad \frac{n-1}{n+1} (1-2d_n)^{n+1} = - \frac{2}{n+1} (1-2d_n)^{n+1}$$

$$- 2d_n(1-2d_n)^n + (1-2d_n)^n .$$

The first term on the right is $O(1/n^{1+2\lambda})$, hence $o(1/n^{1+\lambda})$, and may be ignored. The second term is $O(\log(\frac{n}{\beta})/n^{1+2\lambda})$, hence also $o(1/n^{1+\lambda})$. It remains only to consider the sum of the last term in (4.30) with the last term of (4.31), namely

(4.32) $$(1-2d_n)^n - (1-d_n)^{2n} \quad .$$

Factoring (4.32) as a^n-b^n, it becomes

(4.33) $$- d_n^2 \sum_{i=0}^{n-1} (1-2d_n)^i (1-2d_n+d_n^2)^{n-i-1} \quad .$$

The sum itself is bounded above by $(n-1)(1-d_n)^{2(n-1)} = O(n^{1-2\lambda})$. Thus (4.33) is $O((\log(\frac{n}{\beta}))^2/n^{1+2\lambda}) = o(1/n^{1+\lambda})$ as was to be shown. ||

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Let n arcs, each of length a_n, be placed uniformly at random on the circumference of a circle. If the arc length sequence is chosen so that the coverage probability remains constant, then as n becomes large, a limiting noncentral chi-square distribution with zero degrees of freedom is obtained for the uncovered proportion of the circle. The case of proportionately smaller arcs is also treated, and a limiting normal distribution is found. Applications include immunology, genetics, and time series analysis.

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